SOLUTION OF A NONLINEAR HEAT-CONDUCTION EQUATION FOR VOLUME HEAT SOURCES

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Solutions of a nonlinear heat-conduction equation which are self-similar or self-similar in the limit are discussed for a given total input power.

Solutions of a nonlinear heat-conduction equation which are self-similar or self-similar in the limit are discussed in detail in [1, 2] for a broad class of problems. We present solutions of a similar equation for volume heat sources.

1. We consider a medium whose electrical and thermal conductivities vary as powers of the temperature. We assume the medium is at zero temperature and is placed between two infinite plane electrodes to which a certain potential difference is applied. At time t = 0 a breakdown of the medium occurs over a plane or along a line and the total input power to the medium varies as a power of the time. Then the temperature distribution in the medium will be given by

$$\frac{\partial T}{\partial t} = \frac{a}{r^{\nu-1}} \frac{\partial}{\partial r}\left(r^{\nu-1} \frac{\partial T^n}{\partial r}\right) + AT^m t^p, \tag{1}$$

where $\nu = 1, 2, 3$ according as the problem has plane, axial, or central symmetry. We seek the solution of Eq. (1) for the initial condition T(r, 0) = 0 and the boundary conditions

$$T(r,\ 0) = 0;$$

$$A\varphi(\nu)\int\limits_0^\infty r^{\nu-1}\, T^m t^p\, dr = Q_0 t^\gamma,\ \gamma \geqslant 0,\ \varphi(\nu) = \begin{cases} 2 & \nu = 1, \\ 2\pi & \nu = 2, \\ 4\pi & \nu = 3. \end{cases}$$

It is clear from dimensional considerations that the problem will be self-similar if

$$(1-m)\left[\nu - 2(\gamma+1)\right] = (p+1)\left[(1-n)\nu - 2\right].$$

Then the temperature is given by the expression

$$T = \left(\frac{Q_0}{a^{\frac{\nu}{2}}}\right)^{\frac{2}{2+\nu(n-1)}} t^{\frac{2(\gamma+1)-\nu}{2+\nu(n-1)}} f(\xi),\ \xi = \left[aQ_0^{n-1} t^{n+\gamma(n-1)}\right]^{-\frac{1}{2+\nu(n-1)}} r,$$

where f(ξ) satisfies

$$\frac{d^2 f^n}{d\xi^2} + \frac{\nu-1}{\xi} \frac{df^n}{d\xi} + \frac{n+\gamma(n-1)}{2+\nu(n-1)}\, \xi\, \frac{df}{d\xi} + Bf^m + \frac{\nu - 2(\gamma+1)}{2+\nu(n-1)}\, f = 0, \tag{2}$$

$$B = A\left(\frac{Q_0}{a^{\nu/2}}\right)^{\frac{2(m-1)}{2+\nu(n-1)}}$$

and the boundary conditions

$$f(\infty) = 0,$$

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$$\int_0^\infty \xi^{\nu-1} f^m(\xi)\, d\xi = \frac{1}{\varphi(\nu) B}\,. \tag{3}$$

The problem cannot be solved in general form. We consider the special case of m = 1.

We multiply (2) by $\xi^{\nu-1}$ and integrate from 0 to ∞. If $f(\xi)$ falls off fast enough at infinity we have

$$\lim_{\xi\to 0} \xi^{\nu-1} \frac{df^n}{d\xi} = [A-(\gamma+1)] \int_0^\infty \xi^{\nu-1} f(\xi)\, d\xi,$$

from which

$$A = \gamma + 1\,,$$

if there is no point source of heat at the origin.

Equation (2) is then easily integrated

$$f(\xi) = \begin{cases} \left[\dfrac{n-1}{2n} \dfrac{n+\gamma(n-1)}{2+\nu(n-1)} (\xi_0^2 - \xi^2)\right]^{\frac{1}{n-1}} & \text{for } \xi \leqslant \xi_0, \\ 0 & \text{for } \xi \geqslant \xi_0. \end{cases}$$

The integration constant ξ_0 is found from (3)

$$\xi_0 = \left[\frac{2}{A\varphi(\nu)}\right]^{\frac{n-1}{2+\nu(n-1)}} \left[\frac{n-1}{2n} \frac{n+\gamma(n-1)}{2+\nu(n-1)}\right]^{-\frac{1}{2+\nu(n-1)}} \left[B\left(\frac{\nu}{2}, \frac{n}{n-1}\right)\right]^{\frac{1-n}{2+\nu(n-1)}}.$$

We note that for a linear thermal conductivity (n = 1) the solution has the form

$$f(\xi) = \frac{\exp\left(-\dfrac{\xi^2}{4}\right)}{2^{\nu-1} A\varphi(\nu)\, \Gamma\left(\dfrac{\nu}{2}\right)}\,.$$

2. We now consider the equation

$$\frac{\partial T}{\partial t} = \frac{a}{r^{\nu-1}} \frac{\partial}{\partial r}\left(r^{\nu-1} \frac{\partial T^n}{\partial r}\right) + DT,$$

which corresponds to a constant potential difference between the electrodes, under the conditions

$$T(r, -\infty) = 0;$$

$$D\varphi(\nu) \int_0^\infty r^{\nu-1} T\, dr = Qe^{\alpha t}.$$

The second condition expresses the exponential time increase of the total input power. In this case the solution which is self-similar in the limit has the form

$$T = \left(\frac{Q\alpha^{\frac{\nu-2}{2}}}{a^{\nu/2}} e^{\alpha t}\right)^{\frac{2}{2+\nu(n-1)}} f(\xi), \quad \xi = \left[\frac{\alpha^n}{aQ^{n-1}} e^{\alpha(1-n)t}\right]^{\frac{1}{2+\nu(n-1)}} r.$$

The function $f(\xi)$ is found from the equation

$$\frac{d^2 f^n}{d\xi^2} + \frac{\nu-1}{\xi} \frac{df^n}{d\xi} + \frac{n-1}{2+\nu(n-1)} \xi \frac{df}{d\xi} + \left[\frac{D}{\alpha} - \frac{2}{2+\nu(n-1)}\right] f = 0 \tag{4}$$

and the boundary conditions

$$f(\infty) = 0,$$

$$\frac{D}{\alpha} \varphi(\nu) \int_0^\infty \xi^{\nu-1} f(\xi)\, d\xi = 1.$$

As in Section 1, if there is no point source of heat at r = 0, we obtain $D = \alpha$.

The solution of Eq. (4) will then have the form

$$f(\xi) = \begin{cases} \left\{\dfrac{(n-1)^2}{2n\left[2+\nu(n-1)\right]}(\xi_0^2 - \xi^2)\right\}^{\frac{1}{n-1}} & \text{for } \xi \leqslant \xi_0; \\ 0 & \text{for } \xi \geqslant \xi_0, \end{cases}$$

where

$$\xi_0 = \left[\frac{2}{\varphi(\nu)}\right]^{\frac{n-1}{2+\nu(n-1)}} \left\{\frac{(n-1)^2}{2n\left[2+\nu(n-1)\right]}\right\}^{-\frac{1}{2+\nu(n-1)}} \left[B\left(\frac{\nu}{2}, \frac{n}{n-1}\right)\right]^{\frac{1-n}{2+\nu(n-1)}}.$$

NOTATION

T	is the temperature;
t	is the time;
r	is the linear coordinate;
aT^{n-1}	is the thermal diffusivity;
Qt^{ν}, $Qe^{\alpha t}$	describe the total input power.

LITERATURE CITED

1. G. I. Barenblatt, Prikl. Mathem. i Mekhan., 16, No. 1, 67 (1952).
2. G. I. Barenblatt, ibid, 18, No. 4, 409 (1954).